ST 421 Exam 2 Reference Sheet

*************DISCRETE RANDOM VARIABLES*************

$PMF\ (DRV): P(Y=y) \quad CDF(DRV\&CRV): P(Y<y)$

$PDF(CRV): P(a<Y<b) = CDF_b - CDF_a$

Binomial Dist.:

$X \sim Bin(n,p)$

$X =$ of successes in sample size n

$p = probability\ of\ success$

$pmf: P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$cdf: P(X>x) = \sum_{i=0}^{x} bin(i,n,p)$

$x \in \mathbb{N} \qquad 0 \le p \le 1$

$\mu = np \qquad \sigma^2 = np(1-p)$

Geometric Dist.: find first success (**memoryless**)

$X \sim g(p)$

$X = trial\ number\ of\ 1^{st}\ success$

$p = probability\ of\ success$

$pmf: P(X=x) = (1-p)^{x-1}p$

$cdf: P(X>x) = 1-(1-p)^x$

$x \in \mathbb{N} \qquad 0 \le p \le 1$

$\mu = \frac{1}{p} \qquad \sigma^2 = \frac{1-p}{p^2}$

**special case of Negative Binomial with r = 1.

Hypergeometric Dist.:

$X \sim h(n,r,N)$

Population: N — sample size: n

r successes — (1-r) failures

X: # of successes in sample size n

$pmf: P(X=x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{x}}$

$cdf: P(X>x) =$

$\mu = n\frac{r}{N} \qquad \sigma^2 = n\frac{r}{N}\left(1-\frac{r}{N}\right)\left(\frac{N-n}{N-1}\right)$

**converges to Binomial as r/N -> p

Negative binomial Dist.: trial # of r[th] success

$X \sim nb(r,p)$

$X = trial\ number\ of\ r^{th}\ success$

$p = probability\ of\ success$

$pmf: P(X=x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$

$cdf: P(X>x) =$

$x \in \mathbb{N} \qquad 0 \le p \le 1$

$\mu = \frac{r}{p} \qquad \sigma^2 = \frac{r(1-p)}{p^2}$

Poisson Dist.: number of random occurrences in a specified unit of space or time

$X \sim Poi(\lambda) \qquad \lambda = mean\ process\ rate$

$pmf: P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad 0<\lambda<\infty \qquad x \in \mathbb{N}$

$cdf: P(X \le x) = (see\ table\ for\ \lambda)$

$\mu = \lambda \qquad \sigma^2 = \lambda$

**in Binomial, as $n \to \infty \Rightarrow np \to \lambda$

Tchebysheff's Inequality: $P(|X-\mu| \ge k\sigma) \le \frac{1}{k^2}$

IOW: $P\big((X \le \mu - k\sigma) \cup (X \ge \mu - k\sigma)\big) \le \frac{1}{k^2}$

IOW: $P(|X-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2}$

Variance: $\sigma_X^2 = E[X^2] - (E[X])^2$

$(E[X])^2 = \mu_X^2 \qquad E[X^2] = (\sum xp(X=x))^2$

$E[bX] = bE[X] \qquad \sigma_{bX}^2 = b^2\sigma_X^2$

$E[c] = c \qquad \sigma_c^2 = 0$

$E[a+bX] = a + bE[X] \quad \sigma_{a+bX}^2 = b^2\sigma_X^2$

$E[a+X] = a + E[X] \qquad \sigma_{a+X}^2 = \sigma_X^2$

$E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$

$E[X+Y] = E[X] + E[Y]$

Moments and Moment Generating Functions

$\mu_k' = E(Y^k) \qquad m(t) = E(e^{tY}) \quad \mu_k = E((Y-\mu)^k)$

Find kth derivative of m(t) wrt t. Set t=0. Result is μ_k'.

For DRV: $E[e^{ty}] = \sum e^{ty}p(y)$•For CRV: $E[e^{ty}] = \int e^{ty}p(y)$

Ex$_{DRV}$: $m(t)_{Poisson} = \sum e^{ty}\frac{\lambda^y e^{-\lambda}}{y!} = \sum \frac{(e^t\lambda)^y e^{-\lambda}}{y!} =$

$e^{-\lambda}\sum\frac{(e^t\lambda)^y}{y!} = e^{-\lambda}e^{\lambda e^t}\sum\frac{(e^t\lambda)^y e^{-\lambda e^t}}{y!} = e^{\lambda(e^t-1)} \qquad \ldots$

$m(t)_{Poisson} = e^{\lambda(e^t-1)}$

$m^{(1)}(t)_{Poisson} = \frac{d}{dt}\left(e^{\lambda(e^t-1)}\right) = \left(e^{\lambda(e^t-1)}\right)(e^t\lambda)|_{t=0} = \lambda$

$m^{(2)}(t)_{Poisson} = \frac{d}{dt}\left(\left(e^{\lambda(e^t-1)}\right)(e^t\lambda)\right)$

$= \left(e^{\lambda(e^t-1)}\right)(e^t\lambda)^2 + \left(e^{\lambda(e^t-1)}\right)(e^t\lambda)|_{t=0} = \lambda^2 + \lambda$

Ex$_{CRV}$:

***********CONTINUOUS RANDOM VARIABLES***********

CDF: $F(X) = P(X \le x)$

Properties of CDF: $\lim_{y \to -\infty} F(y) = F(-\infty) = 0$,

$\lim_{y \to +\infty} F(y) = F(+\infty) = 1$, F(y) non-decreasing ie

$y_1 < y_2 \Rightarrow F(y_1) < F(y_2)$, F: right continuous

PDF: $f(y) = \frac{dF(y)}{dy} = F'(y)$, hence, $CDF = \int_{-\infty}^{y} f(t)dt$, and

$P(a \le Y \le b) = \int_a^b f(t)dt$ Properties of PDF:

f(y): nonnegative for all y, $\int_{-\infty}^{+\infty} f(t)dt = 1$

Quantile: $Q_{0.5}$=median, $Q_{0.25}$=1st quantile=25th percentile

Expectation and Variance of a Cont. Rand. Var.

$E[Y] = \int_{-\infty}^{+\infty} yf(y)dy \qquad E[Y^2] = \int_{-\infty}^{+\infty} y^2 f(y)dy$

$V[Y] = E[Y^2] - (E[Y])^2 = E[(Y-E[Y])^2] = \sigma_Y^2$

Uniform Dist.: $Y \sim U(\theta_1, \theta_2)$

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$$f(y) = \begin{cases} \frac{1}{\theta_2-\theta_1} & \theta_1 \le y \le \theta_2 \\ 0 & else \end{cases} \qquad E[Y] = \frac{\theta_1+\theta_2}{2}$$

$$F(y) = \begin{cases} 1 & y > \theta_2 \\ \frac{y-\theta_1}{\theta_2-\theta_1} & \theta_1 \ge y \ge \theta_2 \\ 0 & y \le \theta_1 \end{cases} \qquad V[Y] = \frac{(\theta_2-\theta_1)^2}{12}$$

Normal Dist.: $Y \sim N(\mu, \sigma^2)$

$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, -\infty < y < \infty, and\ \sigma > 0$

Properties of normal pdf: symmetric, bell-shaped, centered at μ. Also, if $-\infty < \mu < \infty, and\ 0 < \sigma^2$, then

$\int_{-\infty}^{+\infty} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = \sqrt{2\pi}\sigma, \quad E[Y] = \mu, \quad V[Y] = \sigma^2$

Standard Normal Dist.: $Z \sim N(0,1)$

CDF of $Z \sim N(0,1)$: $\Phi(z)$

Standardization of Normal variables: If $Y \sim N(\mu, \sigma^2)$, then $Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$

Percentile and Percentage points of *N(0,1)*:

Empirical Rule for Standard Normal Dist.:

$(\mu \pm \sigma)$ contains approx. 68% of the measurements

$(\mu \pm 2\sigma)$: 95%	$(\mu \pm 3\sigma)$: 99.7%
$(\mu + \sigma)$: 34%	$(\mu - \sigma)$: 34%
$(\mu + 2\sigma)$: 47.5%	$(\mu - 2\sigma)$: 47.5%
$(\mu + 3\sigma)$: 49.85%	$(\mu - 3\sigma)$: 49.85%

Approximating the Binomial with the Normal

If $Y \sim Bin(n,p)$ with $np > 10\ and\ n(1-p) > 10 \Rightarrow$

$$P(Y \le y) \approx P\left(Z \le \frac{(y+.5)-np}{\sqrt{np(1-p)}}\right), (Z \sim N(0,1))$$

The Gamma Family of Distributions:

Gamma function (for α>0): $\Gamma(\alpha) = \int_{-\infty}^{+\infty} x^{\alpha-1}e^{-x}dx$

Properties of Gf: For $n \in \mathbb{N}_+$: $\Gamma(n) = (n-1)!$

For any $\alpha > 0$: $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma(1) = 0! = 1$

Gamma Dist.: $Y \sim Gamma(\alpha, \beta) \qquad 0 < y < +\infty$

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)} & y > 0, \ where \\ 0 & else \end{cases}$$

$0 < \alpha < +\infty$ (controls the shape of the distribution), and

$0 < \beta < +\infty$ (controls the scale of the distr.)

Properties of $Y \sim Gamma(\alpha, \beta)$: $E[Y] = \alpha\beta$

$V[Y] = \alpha\beta^2$ mgf: $m(t) = E[e^{ty}] = (1-\beta t)^{-\alpha}$

Exponential Dist. (sp case of Gamma): $Y \sim Exp(\beta)$

Gamma, with $\alpha = 1 \quad (Y \sim Gamma(1, \beta))$

PDF: $f(y) = \frac{1}{\beta}e^{-\frac{y}{\beta}}, \ y > 0, \ \beta > 0$

$E[Y] = \beta$ CDF: $F(y) = P(Y \le y) = 1 - e^{-\frac{y}{\beta}}$

$V[Y] = \beta^2$ memoryless, asymmetric

Chi-Square Dist. (sp case of Gamma): $Y \sim \mathcal{X}(v)$

Gamma, with $\alpha = \frac{v}{2}, \beta = 2$ -> ChiSq with v deg of fr

$v \in \mathbb{N}_+ \quad 0 < y < +\infty$ Properties of ChiSq:

$f(y) = \frac{1}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} y^{\frac{v}{2}-1}e^{-\frac{y}{2}} \qquad E[Y] = v \qquad V[Y] = 2v$

Beta Dist.: $Y \sim beta(\alpha, \beta) \qquad 0 < y < 1$

Complete Beta function:

$$B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$B(r+1, n-r+1) = \frac{1}{(n+1)\binom{n}{r}} \qquad B(.5, .5) = \pi$

$pdf: f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} \quad for\ 0 \le y \le 1 (0\ else)$

$\mu = \frac{\alpha}{\alpha+\beta}\ \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ mgf: dne

coverges to U as a,b ->1 [beta(1,1)=U(0,1)]

$\alpha = \beta: symmetric$

$\alpha < \beta: right-skewed$

$\alpha > \beta: left-skewed$